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Semispaces of Configurations, Cell Complexes of Arrangements

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INTRODUCTION

If $\mathscr{C} = \{P_1, \ldots, P_n\}$ is a numbered configuration of points in the euclidean plane, any subset $\{P_{i_1}, \ldots, P_{i_k}\}$ consisting of all the points of $\mathscr{C}$ lying entirely on one side of a line is called a *semispace* of $\mathscr{C}$. If $\mathscr{C} = \{P_1, \ldots, P_n\}$ and $\mathscr{C}' = \{P'_1, \ldots, P'_n\}$ are two configurations with the property that $\{P_{i_1}, \ldots, P_{i_k}\}$ is a semispace of $\mathscr{C}$ if and only if $\{P'_{i_1}, \ldots, P'_{i_k}\}$ is a semispace of $\mathscr{C}'$, we call $\mathscr{C}$ and $\mathscr{C}'$ *semispace-equivalent.*

On the other hand, if $\mathscr{A} = \{L_1, \ldots, L_n\}$ is a numbered arrangement of lines in the projective plane, it determines a cell complex $\Gamma(\mathscr{A})$ in a natural way; two numbered arrangements $\mathscr{A}$ and $\mathscr{A}'$ whose associated cell complexes $\Gamma(\mathscr{A})$ and $\Gamma(\mathscr{A}')$ are isomorphic, with the isomorphism respecting the numbering of the arrangements, are themselves called *isomorphic.*

Since configurations of points are dual to arrangements of lines, both in terms of incidence relations (i.e., classically) and in terms of their order properties (see [9]), the question arises how these two notions of equivalence are related to each other. Moreover there are several other natural equivalence relations on configurations and arrangements that we might consider as well: for configurations, for example, we might say that $\mathscr{C}$ and

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$\mathscr{C}'$ are "separation-equivalent" if line P_iP_j separates P_k from P_m whenever the same holds for P_i', P_j', P_k', P_m'; or that $\mathscr{C}$ and $\mathscr{C}'$ are "orientation-equivalent" if the orientation of P_i, P_j, P_k is counterclockwise whenever that of P_i', P_j', P_k' is.

In [8] and [9] we introduced the idea of associating to each numbered configuration of n points, as well as to each numbered arrangement of n lines, a circular sequence of permutations of $\{1,..., n\}$, which encodes in combinatorial terms the orientation properties of the configuration or arrangement; this device has since proven fruitful in solving a number of open problems on configurations and arrangements [5, 7, 8, 10, 11, 13, 15, 16, 19, 23]. The same combinatorial tool, which we call an *allowable sequence of permutations*, can be used to study equivalence relations on configurations and arrangements as well, and this is what we do in the present paper. Our results include: (i) a criterion, in terms of their associated sequences of permutations, for two numbered configurations of points to be semispace-equivalent, and a related criterion, also in terms of allowable sequences, for two numbered arrangements of lines (or pseudolines) to give rise to isomorphic cell complexes; (ii) a combinatorial characterization of the cell complexes determined by arrangements of pseudolines, which provides a solution to the problem posed in [17] of extending the characterization suggested by Ringel [21] from simple arrangements to arbitrary ones; (iii) the solution of a discrete version of the isotopy problem (see Section 1); and (iv) the result that if, for a configuration $\{P_1,..., P_n\}$, one knows how many points lie to the left of each directed line P_iP_j, then one can reconstruct precisely *which* ones do. (This last result, which generalizes to higher dimensions, turns out to have extensive ramifications in computational geometry; see [14].)

In Section 1 we show that semispace-equivalence is the appropriate notion for distinguishing those properties of configurations which relate to orientation, separation, and convexity; our chief result there is a characterization (Theorem 1.7 and Corollary 1.10) of semispace-equivalence in terms of a number of equivalent geometric conditions. In Section 2 we show how the cell complexes induced by arrangements of lines, and—more generally—of pseudolines, can be described in terms of their associated allowable sequences, and in particular we show (Theorem 2.9) that for two arrangements to have isomorphic cell complexes means that their associated sequences are related by a somewhat coarser relation than semispace-equivalence, one which we call *local equivalence*. In Section 3 we introduce formally the notion of a *generalized configuration of points*, previously alluded to in [7] and [11], and show how its geometric properties too are described in terms of allowable sequences. This sets the stage for Section 4, where we show that an allowable sequence can be realized geometrically by an arrangement of pseudolines (Theorem 4.1) as well as by a generalized configuration of points (Theorem 4.4), and conclude (Main Result of Sections 2–4) by establishing the three-way connection among (a) allowable sequences modulo local equivalence (respectively, semispace-equivalence), (b) generalized configurations modulo local equivalence (respectively semispace-equivalence), and (c) pseudline arrangements modulo isomorphism (respectively, isomorphism preserving a marked cell). We end the paper (Section 5) with some open problems suggested by this work.

1. Configurations

Recall the following definitions from [9]:

Definition 1.1. Suppose $\mathscr{C}$ is the numbered configuration $\{P_1, ..., P_n\}$ in $\mathbf{E}^2$. Let L be a directed line not orthogonal to any line determined by two members of $\mathscr{C}$, and project $\mathscr{C}$ orthogonally onto L. If we suppress the P's, we get a permutation of $[1, n]$. If we then allow L to turn in the counterclockwise direction, the permutation changes whenever the direction of projection passes through that given by two or more points of $\mathscr{C}$, and we get a periodic sequence of permutations, called *the circular sequence of permutations associated to* $\mathscr{C}$.

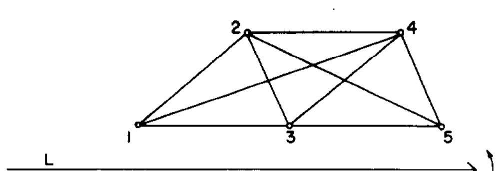
An example is given in Fig. 1, where we have indicated the move from each term of the sequence to the next.

Notice that such a sequence Σ always has the following properties:

(1.1) Σ is periodic;

(1.2) the move from each term of Σ to the next consists of reversing one or more nonoverlapping substrings;

(1.3) if a move results in the reversal of a pair ij then every other pair is reversed subsequently by the time i and j again switch; this guarantees that each period of Σ breaks up into two half-periods, with each move of the first half reversed in the second.



$$\ldots 12345 \xrightarrow{23,45} 13254 \xrightarrow{25} 13524 \xrightarrow{135,24} 53142 \xrightarrow{14} 53412 \xrightarrow{34,12}$$

$$54321 \xrightarrow{54,32} 45231 \xrightarrow{52} 42531 \xrightarrow{42,531} 24135 \xrightarrow{41} 21435 \xrightarrow{21,43} 12345 \ldots$$

Figure 1

DEFINITION 1.2. Any sequence Σ of permutations of $[1, n]$ with properties (1.1), (1.2), and (1.3) is called an *allowable sequence of permutations* of $\{1,..., n\}$, or simply an *n-sequence*. Each *move* of Σ, from one term, Π, to the next, Π', consists of one or more substring reversals, or *switches*, and each of these has an associated *switch symbol*, of the form

$$\frac{i_1 \cdots i_k}{j_1 \cdots j_k},$$

where $i_1,..., i_k$ is a complete substring of Π which is being reversed, and j_p is the position in Π of the index i_p for each p. (Thus the switch symbols $\frac{25}{34}$ and $\frac{531}{345}$ belong to the sequence in Fig. 1.) The *local sequences of ordered switches* of an index i is the periodic sequence of substring reversals that involve i; for example, in Fig. 1 the local sequence of ordered switches belonging to 3 is

$$..., 23, 135, 34, 32, 531, 43, 23,....$$

On the other hand, if we suppress the order in which the members of the successive substrings involving i occur, as well as i itself, we get what is called the *local sequence of unordered switches* of i; for 3 in Fig. 1 it would be

$$..., \{2\}, \{1, 5\}, \{4\}, \{2\}, \{1, 5\},....$$

The reverse $\tilde{\Sigma}$ of a sequence Σ will be the sequence consisting of the same terms, listed in the reverse cyclic order; for example, if Σ is the sequence of Fig. 1 then $\tilde{\Sigma}$ is the sequence

$$...12345 \xrightarrow{12,34} 21435 \xrightarrow{14} 24135 \xrightarrow{24,135} 42531 \xrightarrow{25} 45231 \xrightarrow{45,23}$$

$$54321 \xrightarrow{43,21} 53412 \xrightarrow{41} 53142 \xrightarrow{531,42} 13524 \xrightarrow{52} 13254 \xrightarrow{32,54} 12345....$$

On the other hand, the reverse of a string S of indices will be denoted by $\bar{S}$. We will use the notation $(i_1 j_1 < \cdots < i_r j_r)$ to mean that *within the half-period of Σ following the move in which the indices i_1 and j_1 (in that order) switch*, the indices i_2 and j_2 (in that order) switch in the same or in a subsequent move, then i_3 and j_3, and so on. We emphasize that all of these switches take place within a single half-period. For the sequence of Fig. 1, for example, we have $(43 < 23 < 35 < 24 < 15)$. Notice that for any three indices i, j, k, we always have

$$(ij < jk) \Rightarrow (ij < ik < jk).$$

Finally, we call a sequence *trivial* if all of the indices in it switch simultaneously. (Such a sequence is clearly associated to a collinear configuration of points.)

Remark 1.3. The allowable sequence Σ associated to a configuration $\mathscr{C}$ encodes, in combinatiorial terms, a number of geometric features of $\mathscr{C}$ (let us abbreviate P_i simply as "i"):

(a) $i_1,..., i_k$ are collinear iff they switch simultaneously;

(b) i is in the convex hull $\operatorname{conv}(i_1,..., i_k)$ of $i_1,..., i_k$ iff in every term of Σ, i is preceded by one of $i_1,..., i_k$ (hence "surrounded" by them), and $i_1,..., i_k$ are convexly independent (i.e., none is in the convex hull of the rest) iff each of $i_1,..., i_k$ precedes the rest in some term of Σ;

(c) i is an extreme point of $\mathscr{C}$ iff some term of Σ begins with i, or—equivalently—iff the local sequence of ordered switches belonging to i has the form

$$..., iS_1,..., iS_k, \bar{S}_1 i,..., \bar{S}_k i, iS_1,...;$$

(d) $\overline{ij}$ is parallel to $\overline{km}$ iff i, j switch in the same move (but in disjoint substrings) as k, m;

(e) ijk has positive (i.e., counterclockwise) orientation iff $(ij < ik)$;

(f) line $\overline{ij}$ separates k from m iff when i and j switch, k and m are on opposite sides of the substring containing i and j which reverses; point i separates points j and k (collinear with i) iff when all three switch, i is between j and k;

(g) $\{i_1,..., i_k\}$ constitute a semispace of $\mathscr{C}$ (i.e., all the points of $\mathscr{C}$ lying on one side of a line) iff some term of Σ has $i_1,..., i_k$ (in some order) as an initial segment (hence some other term has them as a terminal segment);

(h) a line L rotating in a counterclockwise direction through point i passes through the remaining points in the cyclic order $..., S_1,..., S_k, S_1,...,$ where each S_j is a subset of C of the form $L \cap (C \backslash i)$, iff the local sequence of unordered switches belonging to i is precisely $..., S_1,..., S_k, S_1,...$;

(i) a directed line L rotating in a counterclockwise direction through point i passes through the points of C in the cyclic order $..., S_1,..., S_k, \bar{S}_1,..., \bar{S}_k, S_1,...,$ where each S_j is a *string* of indices containing i, listed in the order induced by the direction on L, iff the local sequence of ordered switches belonging to i is precisely, $..., S_1,..., S_k, \bar{S}_1,..., \bar{S}_k, S_1,....$

Remark 1.4. Not every allowable sequence arises from a configuration of points (see [8]). Properties (a) through (i) of Remark 1.3 all make sense, however, for an arbitrary allowable sequence, in which the indices $1,..., n$ are regarded as the "points," and so we use them to define the corresponding properties of allowable sequences. Thus, when we speak of a semispace of an n-sequence Σ we shall mean an initial segment in some term of Σ, and so on. Notice, in this connection, that the semispaces of a sequence can be thought of as arising as follows: If the term

$$S_0 i_{11} \cdots i_{1k_1} S_1 i_{21} \cdots i_{2k_2} S_2 \cdots S_{m-1} i_{m1} \cdots i_{mk_m} S_m$$

is transformed into the term

$$S_0 i_{1k_1} \cdots i_{11} S_1 i_{2k_2} \cdots i_{21} S_2 \cdots S_{m-1} i_{mk_m} \cdots i_{m1} S_m$$

by the move

$$\underline{i_{11} \cdots i_{1k_1}, i_{21} \cdots i_{2k_2}, \ldots, i_{m1} \cdots i_{mk_m}},$$

then the following $(k_1 - 1) + \cdots + (k_m - 1)$ new semispaces are "created" by this move:

$$\begin{gathered} S_0 i_{1k_1}, S_0 i_{1k_1} i_{1k_1-1}, \ldots, S_0 i_{1k_1} i_{1k_1-1} \cdots i_{12};\ S_0 i_{1k_1} \cdots i_{11} S_1 i_{2k_2}, \\ S_0 i_{1k_1} \cdots i_{11} S_1 i_{2k_2} i_{2k_2-1}, \ldots, S_0 i_{1k_1} \cdots i_{11} S_1 i_{2k_2} i_{2k_2-1} \cdots i_{22}; \\ \cdots \\ S_0 \cdots S_{m-1} i_{mk_m}, S_0 \cdots S_{m-1} i_{mk_m} i_{mk_m-1}, \ldots, S_0 \cdots S_{m-1} i_{mk_m} \cdots i_{m2}. \end{gathered}$$

The move

$$13524 \xrightarrow{135,24} 53142$$

in the sequence of Fig. 1, for example, creates the semispaces

$$\{5\}, \{5, 3\}, \{5, 3, 1, 4\}.$$

DEFINITION 1.5. An *elementary transformation* of one allowable sequence into another consists either of amalgamating two successive moves whose substrings undergoing reversal are disjoint, or of the reverse, i.e., separating a move into two successive moves; in each case of course the corresponding amalgamation or separation must take place in every half-period of the sequence. For example, we may apply an elementary transformation to the 6-sequence

$$\ldots 123456 \xrightarrow{34} 124356 \xrightarrow{124} 421356 \xrightarrow{56} 421365 \xrightarrow{136} 426315 \xrightarrow{26,15} 462351$$
$$\xrightarrow{46,235} 645321 \xrightarrow{45} 654321 \xrightarrow{43} 653421 \xrightarrow{421} 653124 \xrightarrow{65} 563124 \xrightarrow{631} \ldots$$

to get the 6-sequence

$$\ldots 123456 \xrightarrow{34} 124356 \xrightarrow{124,56} 421365 \xrightarrow{136} 426315 \xrightarrow{26,15} 462351$$

$$\xrightarrow{46,235} 645321 \xrightarrow{45} 654321 \xrightarrow{43} 653421 \xrightarrow{65,421} 563124 \xrightarrow{631} \ldots.$$

Hence by a succession of two elementary transformations we can interchange two successive moves involving disjoint sets of indices.

Remark 1.6. Theorem 1.7, which follows, gives a number of conditions on a pair Σ, Σ' of allowable sequences (hence of configurations), both "geometric" and combinatorial, which are equivalent to the statement that Σ and Σ' have the same sets of semispaces. Several of these conditions are worth commenting on. (i) says that Σ and Σ' have the same associated (rank 3) *oriented matroid* in the sense of [2] or [6]. (ii) says they have the same *separation function* in the sense of [24]. (iii) says they have the same associated affine *chirotope*, in the sense of [4]. (iv) says they have the same contracted (rank 2) oriented matroid for each point i, or—in the language of [4]—the same spherical chirotope restriction for each i. Perhaps the most striking of these results are (vii) $\Rightarrow$ (iii), which yields Corollary 1.11 (see below), and (i) $\Rightarrow$ (vi), which says that if Σ and Σ' are two such sequences then Σ' can be obtained from Σ by a sequence of elementary transformations. This can be viewed as a discrete version of the (as yet unproven) isotopy conjecture for configurations, which asks for a continuous transformation of a numbered configuration $\mathscr{C}$ of points into another, $\mathscr{C}'$, with the same set of semispaces, with each intermediate configuration also having the same set of semispaces (see Section 5 and [15] for further comments on this problem).

THEOREM 1.7. *Let $\Sigma = (\Pi_i)_{i\in\mathbf{Z}}$ and $\Sigma' = (\Pi'_i)_{i\in\mathbf{Z}}$ be two nontrivial allowable n-sequences. Then the following are equivalent*:

(i) *Σ and Σ' have the same sets of semispaces*;

(ii) *line $\overline{jk}$ separates i from m in Σ iff the same holds in Σ'*;

(iii) *each noncollinear triple ijk has the same orientation in Σ as in Σ' (or else this holds for the triples of Σ and $\tilde{\Sigma}'$)*;

(iv) *Σ and Σ' (or else $\tilde{\Sigma}'$) have the same local sequence of ordered switches for each i, $1 \leqslant i \leqslant n$*;

(v) *Σ and Σ' (or else $\tilde{\Sigma}'$) have the same local sequence of unordered switches for each i, $1 \leqslant i \leqslant n$*;

(vi) *Σ can be transformed into Σ' (or else into $\tilde{\Sigma}'$) by a sequence of elementary transformations*;

(vii) *the set of switch symbols in Σ is the same as that in Σ' (or else in $\tilde{\Sigma}'$).*

Proof. We shall prove the implications

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(vi)} \Rightarrow \text{(vii)} \Rightarrow \text{(i)};$$

the equivalence of (iv) and (v) will be shown later, in Section 4, Corollary 4.2.

(i) $\Rightarrow$ (ii): Just observe that $\overline{jk}$ separates i from m iff there are semispaces whose intersections with the set $\{i, j, k, m\}$ are $\{i\}$, $\{i, j\}$, $\{i, k\}$, $\{i, j, k\}$, and their complements: the direct implication is immediate, while the converse follows from an enumeration of all possible 4-sequences.

(ii) $\Rightarrow$ (iii): Suppose 123 is positive, i.e., $(12 < 13)$, in both sequences. (If not, replace Σ' by $\tilde{\Sigma}'$.) Assume xyz is a noncollinear triple. Then one of $12x$ or $12y$ or $12z$ must be noncollinear, say, $12z$. Then, similarly, $1xz$ or $1yz$ must be noncollinear, say, $1yz$. In that case, according as $\overline{12}$ separates 3 from z, or not, we have $(1z < 12)$ (resp. $(12 < 1z)$), i.e., $12z$ is negative (resp. positive) in both sequences. Similarly, $1yz$ agrees in both, and finally xyz does.

(iii) $\Rightarrow$ (iv): Say Σ and Σ' (rather than $\tilde{\Sigma}'$) have the same orientations of triples. If we know which triples ijk have positive orientation, then we know which ones are collinear (those having no positive permutations!). Consider the local 1-sequence. Partition the points $2,..., n$ by the equivalence relation

$$i \sim j \qquad \text{iff } 1, i, j \text{ are collinear,}$$

and choose a complete set of representatives $i_1,..., i_k$. Then first of all, knowing the orientation of each triple $1, i_p, i_q$, i.e., knowing whether

$$(1i_p < 1i_q) \qquad \text{or} \qquad (1i_q < 1i_p),$$

we can reconstruct uniquely the circular order of all the symbols $1i_1,..., 1i_k$, $i_1 1,..., i_k 1$. Moreover, the order of the points $1, i_p, j_1,..., j_r$, where $i_p, j_1,..., j_r$ constitute an equivalence class, is also determined: just look at any point j_0 *not* collinear with them, and consider the orientation of all the triples involving j_0 and two of the points of the collinear set. The assertion follows.

(iv) $\Rightarrow$ (vi): We shall prove first that if Σ and Σ' have the same local sequences of ordered switches then a succession of elementary transformations can be applied to each so that the resulting sequences have a common term. (Let us remark first that it is possible for two sequences satisfying condition (iv) to have *no* terms in common, even in the "simple" case (i.e., no multiple or simultaneous switches):

$$\ldots 123456 \xrightarrow{12} 213456 \xrightarrow{34} 214356 \xrightarrow{14} 241356 \xrightarrow{24} 421356 \xrightarrow{35} 421536$$
$$\xrightarrow{36} 421563 \xrightarrow{15} 425163 \xrightarrow{16} 425613 \xrightarrow{25} 452613 \xrightarrow{26} 456213 \xrightarrow{45} 546213$$
$$\xrightarrow{46} 564213 \xrightarrow{13} 564231 \xrightarrow{56} 654231 \xrightarrow{23} 654321 \ldots$$

and

$$\ldots 124356 \xrightarrow{35} 124536 \xrightarrow{36} 124563 \xrightarrow{12} 214563 \xrightarrow{14} 241563 \xrightarrow{15} 245163$$
$$\xrightarrow{16} 245613 \xrightarrow{13} 245631 \xrightarrow{24} 425631 \xrightarrow{25} 452631 \xrightarrow{26} 456231 \xrightarrow{23} 456321$$
$$\xrightarrow{45} 546321 \xrightarrow{46} 564321 \xrightarrow{43} 563421 \xrightarrow{56} 653421 \ldots$$

are two such sequences.)

Now suppose Σ and Σ' have the same local sequences of ordered switches; by applying elementary transformations, if necessary, we may assume each move of Σ and Σ' is a switch of a single string. Say 1 is an extreme point in Σ. Then its local sequence has the form

$$\ldots,\ 1S_1,\ 1S_2, \ldots,\ 1S_p,\ \bar{S}_1 1,\ \bar{S}_2 1, \ldots,\ \bar{S}_p 1,\ 1S_1, \ldots,$$

where $S_1, \ldots, S_p$ are strings whose disjoint union is $\{2, \ldots, n\}$, and $\bar{S}_i$ is the reverse of S_i. Hence this is the local sequence of 1 in Σ', so that in particular 1 is also an extreme point of Σ'. By a succession of elementary transformations, the moves in Σ not involving 1 which intervene among $1S_1, \ldots, 1S_p$ can be moved either to the left of $1S_1$ or to the right of $1S_p$, as follows: any switch involving only indices $\in S_1 \cup \cdots \cup S_k$ which occurs after S_k can be moved to the *right*, past $1S_p$, while any switch involving only indices $\in S_m \cup \cdots \cup S_p$ which occurs before S_m can be moved to the *left*, past $1S_1$. How can we get stuck? Only if we have a switch involving indices i and j, with $i \in S_k$ and $j \in S_m$, which occurs *between* $1S_k$ and $1S_m$. But then we would have, supposing $k < m$:

$$(1i < ij < 1j) \qquad \text{or} \qquad (1i < ji < 1j),$$

both of which are impossible (see Definition 1.2). Hence by a succession of elementary transformations Σ (resp. Σ' or $\tilde{\Sigma}'$) can be changed into a new sequence Σ_1 (resp. Σ_1') in which the moves

$$1S_1,\ 1S_2, \ldots,\ 1S_p$$

occur in that order with no other moves intervening. But then in each of Σ_1

and Σ'_1 the term immediately preceding these moves must be $1S_1S_2 \cdots S_p$, so that we have produced a common term.

We now proceed, by elementary transformations, to create more and more common terms without losing any of the old ones. Suppose at some stage Σ (resp. Σ' or $\tilde{\Sigma}'$) has been transformed into a sequence Σ_r (resp. Σ'_r) such that Σ_r and Σ'_r each contain the terms Π^* and Π^{**}, with Π^{**} occurring no more than a half-period later than Π^* in each. (Of course Σ_r and Σ'_r still have the same local sequences of ordered switches, since an elementary transformation has no effect on the set of local sequences.) Let the intervening moves and terms be as follows:

$$\Sigma_r = \dots \Pi^* = \Pi_1 \xrightarrow{M_1} \Pi_2 \xrightarrow{M_2} \dots \xrightarrow{M_{p-1}} \Pi_p = \Pi^{**} \dots$$

$$\Sigma'_r = \dots \Pi^* = \Pi'_1 \xrightarrow{M'_1} \Pi'_2 \xrightarrow{M'_2} \dots \xrightarrow{M'_{q-1}} \Pi'_q = \Pi^{**} \dots.$$

We claim first that the sets $\{M_1, \dots, M_{p-1}\}$ and $\{M'_1, \dots, M'_{q-1}\}$ are equal: this is clear, since—as a result of the assumed equality of the local sequences in Σ_r and in Σ'_r—the set of *all* moves in Σ_r equals that in Σ'_r, and since precisely the same pairs i, j are switched as a result of the moves $M_1, \dots, M_{p-1}$ as are switched as a result of the moves $M'_1, \dots, M'_{q-1}$—namely, each pair whose position in Π^{**} is the reverse of its position in Π^*. Now consider M_1. If this is the same as M'_1, then necessarily $\Pi_2 = \Pi'_2$. If not, say, $M_1 = M'_t$ for some $1 < t < q$. We claim that no index i appearing in M'_t can occur in any M'_s for $1 \leqslant s < t$: the reason is that otherwise i would be involved in two distinct moves (M'_s and M'_t) which occurred in Σ_r and Σ'_r *in the opposite order*, and in each case within a portion of the sequence *no more than a half-period in length*, and this would violate the supposed equality of the local sequences belonging to i in Σ_r and Σ'_r. Hence by a succession of elementary transformations, starting with Σ'_r, M'_t can be moved to the left, to yield the sequence

$$\Sigma'_{r+1} = \dots \Pi^* = \Pi'_1 \xrightarrow{M'_t} \Pi_2 = \Pi''_2 \xrightarrow{M'_1} \Pi''_3 \xrightarrow{M'_2} \dots$$
$$\xrightarrow{M'_{t-1}} \Pi''_{t+1} \xrightarrow{M'_{t+1}} \dots \xrightarrow{M'_{q-1}} \Pi'_q = \Pi^{**} \dots,$$

and induction (on the number of common terms) then yields the result.

(vi) $\Rightarrow$ (vii): Since in the amalgamation of two successive moves involving disjoint sets of indices these indices necessarily occupy disjoint sets of positions, and since they occupy the same positions in the resulting move, it is clear that an elementary transformation preserves the set of switch symbols; the result follows.

(vii) $\Rightarrow$ (i): Since Σ' and $\tilde{\Sigma}'$ always have the same semispaces, we may assume that Σ and Σ' have the same sets of switch symbols. First notice that an elementary transformation of an allowable sequence Σ has no effect on the set of switch symbols of Σ ((vi) $\Rightarrow$ (vii)), or on the set of semispaces of Σ (if $\Pi_1 \xrightarrow{M_1} \Pi_2 \xrightarrow{M_2} \Pi_3$ is amalgamated into $\Pi_1 \xrightarrow{M_1, M_2} \Pi_3$, or the latter separated into the former, where M_1 and M_2 involve disjoint sets of indices, every initial and terminal segment of Π_2 is already present in either Π_1 or Π_3). Hence we may assume that each move of Σ and Σ' consists of a single switch. The assertion will therefore follow from the stronger statement: If Σ and Σ' are (possibly trivial) n-sequences without simultaneous switches having the same sets of switch symbols then each move creates the same set of semispaces in Σ' as it does in Σ (see Remark 1.4).

We use induction on n. If $n = 3$, the only possibilities for Σ or Σ' are

$$\ldots 123 \frac{12}{12} 213 \frac{13}{23} 231 \frac{23}{12} 321 \frac{21}{23} 312 \frac{31}{12} 132 \frac{32}{23} 123\ldots,$$

$$\ldots 123 \frac{12}{12} 132 \frac{13}{12} 312 \frac{12}{23} 321 \frac{32}{12} 231 \frac{31}{23} 213 \frac{21}{12} 123\ldots,$$

$$\ldots 123 \frac{123}{123} 321 \frac{321}{123} 123\ldots, \ldots 231 \frac{231}{123} 132 \frac{132}{123} 231\ldots,$$

and

$$\ldots 312 \frac{312}{123} 213 \frac{213}{123} 312\ldots,$$

and Σ and Σ' must be the same sequence for the hypothesis to hold. Suppose $n > 3$. By renumbering if necessary, we may assume n is an extreme point of Σ (it is obvious that any n-sequence for $n > 1$, even a trivial one, has at least two extreme points). Then as a result of Remark 1.3 (c), the switch symbols involving n in Σ (hence in Σ') must have the form

$$\frac{nS_1}{1 \cdots k_1}, \frac{nS_2}{k_1 + 1 \cdots k_2}, \ldots, \frac{nS_r}{k_{r-1} + 1 \cdots k_r}, \frac{\bar{S}_1 n}{n - k_1 + 1 \cdots n}, \ldots, \tag{1.4}$$
$$\frac{\bar{S}_r n}{n - k_r + 1 \cdots n - k_r - 1},$$

where $k_r = n$, and the sets $S_1, \ldots, S_r$ form a partition of $[1, n-1]$.

Let Σ_n and Σ_n' be the sequences obtained from Σ and Σ' by deleting n from each term and removing any repeated terms that result. Because (1.4) is actually the local sequence belonging to n in each of Σ and Σ', we can tell whether n is to the left or the right of the pair j, k when the switch $...j...k...$ occurs: if $j \in S_{i_j}$ and $k \in S_{i_k}$ with $i_j < i_k$ we have

$$(nj < nk) \Rightarrow (nk < jn) \Rightarrow (nk < jk < jn)$$

(see Definition 1.2), so that n must be to the *right* of j and k when they switch; of course if $S_{i_j} = S_{i_k}$ then j, k, and n all switch simultaneously, so the corresponding switch symbol tells us the position of each. Hence the switch symbol of each move of Σ_n is the same as that of the corresponding move of Σ'_n, so that by induction hypothesis each move of Σ_n creates the same sets of semispaces as the corresponding move of Σ'_n.

It remains to show that this remains true when we reinsert n. Suppose first that a move

$$i_1 \cdots i_j i_{j+1} \cdots i_k i_{k+1} \cdots i_n \xrightarrow{i_{j+1} \cdots i_k} i_1 \cdots i_j i_k \cdots i_{j+1} i_{k+1} \cdots i_n$$

of Σ does not involve n. The semispaces created by this move are

$$\{i_1,..., i_j, i_k\}, \{i_1,..., i_j, i_k, i_{k-1}\},..., \{i_1,..., i_j, i_k,..., i_{j+2}\}.$$

If n is among $i_1,..., i_j$ (resp. $i_{k+1},..., i_n$) in Σ, then the positions of $i_{j+1},..., i_k$ drop by one (resp. remain the same) in Σ_n, hence also in Σ'_n, hence n is to the left (resp. right) of $i_1,..., i_j$ when these switch in Σ'. But in the term of Σ' following the move $i_{j+1} \cdots i_k$, i_k *is to the left of* i_{j+1}, so we can identify those semispaces created by the move $i_{j+1} \cdots i_k$ in Σ'_n to which n should be added to get the corresponding semispaces of Σ', and they are the same as the ones to which n should be added when we go from Σ_n to Σ. Finally, since it is clear that for any p, $1 \leqslant p \leqslant r$, the move

$$nS_p = ni_1 \cdots i_j$$

creates precisely the same sets of semispaces in Σ and in Σ', namely,

$$\Big(\bigcup_{i<p} S_i\Big) \cup \{i_j\}, \Big(\bigcup_{i<p} S_i\Big) \cup \{i_j, i_{j-1}\},...,$$
$$\Big(\bigcup_{i<p} S_i\Big) \cup \{i_j,..., i_1\},$$

the assertion is proven.

DEFINITION 1.8. If Σ and Σ' are related as in Theorem 1.7, they are called *semispace-equivalent*.

Remark 1.9. It is natural to think of two numbered configurations as *combinatorially equivalent* if they give rise to the same allowable sequence of permutations; thus semispace-equivalence is weaker than combinatorial equivalence, and in particular the allowable sequence structure on configurations can be thought of as a refinement of their oriented matroid structure. Figure 2 shows that it is indeed a proper refinement: the two numbered configurations shown are semispace-equivalent, but not combinatorially equivalent.

COROLLARY 1.10. *Let* $\mathscr{C} = \{P_1, \ldots, P_n\}$ *be a noncollinear configuration in* $\mathbf{E}^2$, *and let* $\mathscr{C}' = \{P'_1, \ldots, P'_n\}$ *be another. Then the following are equivalent*:

(i) $\{P_{i_1}, \ldots, P_{i_k}\}$ *is a semispace of C iff* $\{P'_{i_1}, \ldots, P'_{i_k}\}$ *is a semispace of* $\mathscr{C}'$;

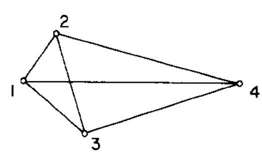
(ii) *line* P_jP_k *separates points* P_i *and* P_m *iff line* $P'_jP'_k$ *separates points* P'_i *and* P'_m;

(iii) $(P_{i_1}, P_{i_2}, P_{i_3})$ *has the same orientation as* $(P_{j_1}, P_{j_2}, P_{j_3})$ *iff* $(P'_{i_1}, P'_{i_2}, P'_{i_3})$ *has the same orientation as* $(P'_{j_1}, P'_{j_2}, P'_{j_3})$;

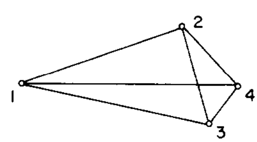
(iv) *for each* i, $1 \leqslant i \leqslant n$, *a directed line* L *rotating through point* P_i *in a counterclockwise direction passes through the points of* $\mathscr{C}$ *in the cyclic order* $\ldots, T_1, \ldots, T_m, T_1, \ldots$, *where each* T_j *is a subset of* $\mathscr{C}$ *of the form* $L \cap \mathscr{C}$ *with an ordering induced by the direction on* L, *iff a directed line* L' *rotating through point* P'_i *in a counterclockwise direction passes through the points of* $\mathscr{C}'$ *in the cyclic order* $\ldots, T'_1, \ldots, T'_m, T'_1, \ldots$ (*with* T'_j *corresponding to* T_j), *where each* T'_j *is a subset of* $\mathscr{C}$ *of the form* $L' \cap \mathscr{C}'$ *with an ordering induced by the direction on* L' (*or else this is true with the second "counterclockwise" replaced by "clockwise"*);

(v) *the same as* (iv), *except that* L *and* L' *are ordinary* (*not directed*) *lines, and that* $T_1, \ldots, T_m, T'_1, \ldots, T'_m$ *are ordinary* (*not ordered*) *subsets.*

Proof. This follows immediately from Theorem 1.7 and Remark 1.3.



$\ldots\ 1234 \underline{\ 23\ } 1324 \underline{\ 13\ } 3124 \underline{\ 24\ } 3142 \underline{\ 14\ } 3412 \underline{\ 34\ } 4312 \underline{\ 12\ } 4321 \ldots$



$\ldots\ 1234 \underline{\ 23\ } 1324 \underline{\ 24\ } 1342 \underline{\ 13\ } 3142 \underline{\ 14\ } 3412 \underline{\ 12\ } 3421 \underline{\ 34\ } 4321 \ldots$

FIGURE 2

We state the following as a separate corollary, rather than as part of Corollary 1.10, because of its striking nature:

COROLLARY 1.11. *Let* $\mathscr{C} = \{P_1, \ldots, P_n\}$ *and* $\mathscr{C}' = \{P'_1, \ldots, P'_n\}$ *be configurations in* $\mathbf{E}^2$. *If, for every* $i \neq j$, *the* number *of points of* $\mathscr{C}$ *to the left of the directed line* $\overrightarrow{P_iP_j}$ *is the same as the number of points of* $\mathscr{C}'$ *to the left of* $\overrightarrow{P'_iP'_j}$, *then, for every* $i \neq j$, *the* set *of points of* $\mathscr{C}$ *to the left of* $\overrightarrow{P_iP_j}$ *is the "same" as the set of points of* $\mathscr{C}'$ *to the left of* $\overrightarrow{P'_iP'_j}$.

Proof. If the points of $\mathscr{C}$ or of $\mathscr{C}'$ are all collinear, the assertion is trivially true. Let us therefore suppose that this is not the case. Then the associated n-sequences, Σ and Σ' resp., are nontrivial. The result will now follow from (vii) $\Rightarrow$ (iii) of Theorem 1.7, as soon as we interpret the hypothesis in terms of switch symbols. Suppose $\overrightarrow{ij}$ has as many points on its left in $\mathscr{C}$ as in $\mathscr{C}'$. Then in the sequence Σ of $\mathscr{C}$, when we project in the direction of $\overrightarrow{ij}$ the points on the left of the line $\overrightarrow{ij}$ show up to the right of the indices i, j in the terms immediately preceding and following the move in which i and j switch (this is an anomaly of our method of forming sequences!). Hence the switch symbol involving the pair i, j in Σ has the same sets of positions in the lower half of the symbol as in Σ'; it remains to show that each i itself occupies the same position in them.

Now an *edge* consists of a maximal set $\{i_1, \ldots, i_p\}$ such that each line $\overrightarrow{i_ji_k}$ $(1 \leqslant j < k \leqslant p)$ has nothing to its *right*. One can tell the *order* of the points on the edge, since there is no other order in which this will be the case. So in particular one can tell the *extreme* points. (And they are the same in $\mathscr{C}$ as in $\mathscr{C}'$.)

Let i be an extreme point and imagine a ray R (resp. R') swinging counterclockwise around i, starting pointing away from conv$(1, \ldots, n)$ in $\mathscr{C}$ (resp. $\mathscr{C}'$). If R hits the remaining points of $\mathscr{C}$ in the order $S_1, \ldots, S_q$, where each S_j is a complete nonempty set of the form $R \cap \mathscr{C} \backslash \{i\}$, and hits $\mathscr{C}'$ in the order $S'_1, \ldots, S'_r$, where each S'_i is a complete nonempty set of the form $R \cap \mathscr{C}' \backslash \{i\}$, then it is clear—by induction—that S_j must equal S'_j, and r must equal q—otherwise the hypothesis would be violated at some stage. It follows that the order of the points on any $\overrightarrow{ij}$ is determined—just look at them from some extreme point off the line!

Remark 1.12. Corollary 1.11, as well as its generalization to higher dimensions, turns out to have extensive ramifications in computational geometry and several other applied areas—see [14].

As a further corollary, assuming the truth of Theorem 1.7, (v) $\Rightarrow$ (iv), for the moment, we get a new proof of [9, Theorem 1.4]:

COROLLARY 1.13. *If* Σ *and* Σ' *are n-sequences with the* same *(global) sequence of unordered switches, then* $\Sigma = \Sigma'$.

Proof. By Theorem 1.7, (v) $\Rightarrow$ (iv), Σ and Σ' have the same local sequences of *ordered* switches, for each i. In order to see that they have the same *global* sequence of ordered switches, it remains to show that for any four distinct indices, i, j, k, m, we have

$$(ij \prec km) \text{ in } \Sigma \Rightarrow (ij \prec km) \text{ in } \Sigma'.$$

If not, then we would have

$$(ij \prec mk) \qquad \text{in } \Sigma'.$$

Now in Σ, we have either

$$(ij \prec ik \prec km) \quad \text{or} \quad (ij \prec km \prec ik) \quad \text{or} \quad (ij \prec ki \prec km)$$
$$\text{or} \quad (ij \prec km \prec ki).$$

Hence, by hypothesis, we would have in Σ' (respectively):

$$(ij \prec ik \prec mk) \quad \text{or} \quad (ij \prec mk \prec ik) \quad \text{or} \quad (ij \prec ki \prec mk)$$
$$\text{or} \quad (ij \prec mk \prec ki),$$

which would contradict the fact that the local k-sequence of ordered switches is supposed to agree in Σ and in Σ'. But now it is immediate that each term Π is determined by the half-period of ordered switches immediately following it; namely, if $\Pi = i_1 \cdots i_n$, the pairs i_j, i_k with $j < k$ must be precisely the ones undergoing reversal in that half-period!

A related result is the following, whose geometric content is that the set of permutations of $[1, n]$ obtained by projecting a configuration $P_1, \ldots, P_n$ in all directions determines the configuration up to equivalence:

COROLLARY 1.14. *If Σ and Σ' are n-sequences with the same sets of* terms *then $\Sigma' = \Sigma$ (or $\tilde{\Sigma}$).*

Proof. By renumbering if necessary, let us assume that one of the terms is $123\ldots n$. Then the next term in Σ must be one of the two (at most) in which the number of reversals (i.e., pairs ij with $i < j$ and j occurring before i) is minimum. Choosing one of these, we can then proceed similarly, and in a unique manner, to reconstruct the rest of Σ. Similarly, we determine Σ'. The result must clearly be either Σ or its reverse.

Finally, as for convexity properties, we have:

PROPOSITION 1.15. *If*

(i) *Σ and Σ' are semispace-equivalent then*

(ii) $i \in \operatorname{conv}_\Sigma(j_1, \ldots, j_k) \Leftrightarrow i \in \operatorname{conv}_{\Sigma'}(j_1, \ldots, j_k)$.

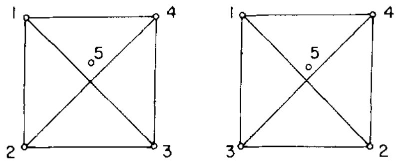


FIGURE 3

Furthermore, (ii) *implies*

(iii) $j_1,\ldots,j_k$ *are convexly independent in* $\Sigma \Leftrightarrow$ *they are in* Σ'.
(*However,* (ii) $\Rightarrow$ (i) *and* (iii) $\Rightarrow$ (ii) *are both false.*)

Proof. (i) $\Rightarrow$ (ii) follows immediately from condition (i) of Theorem 1.7, since $i \in \mathrm{conv}_\Sigma(j_1,\ldots,j_k)$ if and only if every semispace of Σ which contains i contains one of $j_1,\ldots,j_k$; (ii) $\Rightarrow$ (iii) is obvious. As for the inverse implications, their failure is seen in the examples of Figs. 3 and 4, respectively.

2. ARRANGEMENTS

Associated to an arrangement $\mathscr{A}$ of n lines in $\mathbf{P}^2$ there are also allowable n-sequences. These are obtained as follows [9]:

Choose a line L_∞ not in $\mathscr{A}$ to play the role of the "line at infinity," and a point P on $L_\infty \backslash \bigcup_{L \in \mathscr{A}} L$ to play the role of the "vertical point at infinity." We may then identify $\mathscr{A}$ with an arrangement of lines in $\mathbf{E}^2$, none of which is vertical. Choose a "horizontal" direction, and call it "left-to-right." Now let a vertical line L sweep across $\mathscr{A}$ from left to right, and read off the permutations of $[1, n]$ obtained by reading its intersections with the members of $\mathscr{A}$ from top to bottom, say. It is clear that this gives a half-period of an allowable sequence, so we may complete it uniquely to a full n-sequence; this could equally well be achieved by sweeping across a second time, and reading from bottom to top. This procedure is shown in Fig. 5, from which the sequence obtained is the same as that coming from the configuration in Fig. 1.

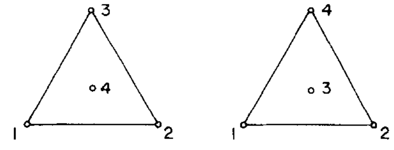


FIGURE 4

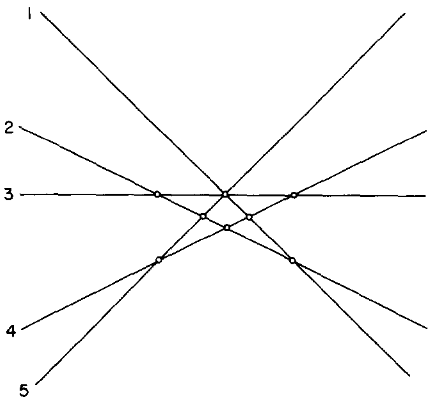


FIGURE 5

Various geometric properties of $\mathscr{A}$ are also reflected in a sequence coming from $\mathscr{A}$; however, we first wish to apply this procedure to a more general situation before discussing the geometric aspect. To this end, let us recall from [17] that an arrangement $\mathscr{A}$ of pseudolines in $\mathbf{P}^2$ is a family of non-contractible simple closed curves, any two of which meet at only one point. Thus each $L \in \mathscr{A}$ is the image of a straight line under some homeomorphism of $\mathbf{P}^2$ with itself, and the same clearly holds for any two pseudolines in an arrangement. It holds, in fact, for any *eight* pseudolines [10], but not for any nine [18, 21]. Thus there are pseudoline arrangements which are not "stretchable" in this sense, so that pseudoline arrangements form a proper generalization of straight line arrangements, even from the topological point of view. (They do not, however, constitute a "generalization for the sake of generalization": indeed, we shall see later that they arise naturally as the geometric objects which represent the combinatorial objects we use to classify line arrangements.) It is easy to see, however, by a simple topological argument, that any "pencil of pseudolines," i.e., arrangement in which all pass through a common point, *is* stretchable; we shall use this fact below.

There are a few facts we shall need about pseudoline arrangements which we recall now: If $\mathscr{A}$ is a pseudoline arrangement in $\mathbf{P}^2$, its *induced cell complex* $\Gamma(\mathscr{A})$ consists of the connected components of $\mathbf{P}^2 \backslash \bigcup_{L \in \mathscr{A}} L$ as the 2-cells, the connected components of $L_i \backslash \bigcup_{j \neq i} L_j$ for all i as the 1-cells, and the vertices $L_i \cap L_j$ as the 0-cells, together with their incidence relations. It is easy to see that any isomorphism $f: \Gamma(\mathscr{A}) \to \Gamma(\mathscr{A}')$ of the cell complexes arising from two numbered arrangements induces a 1–1 correspondence between the members of $\mathscr{A}$ and those of $\mathscr{A}'$; if this correspondence respects the numbering of the members of $\mathscr{A}$ and $\mathscr{A}'$, i.e., if—for every 1-cell $\sigma \in \Gamma(\mathscr{A})$—

$$\sigma \subset L_i \Leftrightarrow f(\sigma) \subset L_i',$$

we call f a *labelled isomorphism* of the arrangements $\mathscr{A}$ and $\mathscr{A}'$. (Since all of our isomorphisms will be labelled ones, we shall just refer to this as an *isomorphism* in the sequel.) It follows from the Schoenflies theorem [20] that $\mathscr{A}$ is isomorphic to $\mathscr{A}'$ if and only if there is a homeomorphism $\alpha: \mathbf{P}^2 \to \mathbf{P}^2$ such that $\alpha(L_i) = L_i'$ for all i. Thus to say, as above, that an arrangement $\mathscr{A}$ is stretchable means precisely that it is isomorphic to an arrangement of straight lines.

PROPOSITION 2.1. *Let $\mathscr{A}^* = \{L_1,..., L_{n+1}\}$ be a pseudoline arrangement which is not a pencil, let $\mathscr{A}$ be the subarrangement $\{L_1,..., L_n\}$, and let σ be any (open) 2-cell of $\mathscr{A}$. If L_{n+1} meets the closure $\bar{\sigma}$ of σ, then either L_{n+1} meets the boundary of σ in precisely two points which it joins by an arc lying entirely in σ, or else it meets $\bar{\sigma}$ at precisely one point, namely, at a vertex of σ.*

This is fairly clear from the definition of an arrangement; we refer the reader to [18, p. 261] for the details.

PROPOSITION 2.2 (Levi Enlargement Lemma). *If $\mathscr{A}$ is an arrangement of pseudolines and P, Q are two points not both lying on any member of $\mathscr{A}$, there is a pseudoline L passing through P and Q such that $\mathscr{A} \cup \{L\}$ is still an arrangement.*

For the proof, see [17] or [18].

If $\mathscr{A}$ is a numbered arrangement of pseudolines we can assign to it a sequence of permutations in much the same way as to an arrangement of lines, with the help of the Levi enlargement lemma:

DEFINITION 2.3. Let $\mathscr{A} = \{L_1,..., L_n\}$ be an arrangement of pseudolines in $\mathbf{P}^2$, and σ any 2-cell belonging to $\mathscr{A}$. (The pair $(\mathscr{A}, \sigma)$ will be called a *marked arrangement*.) Let P be any point in σ. By repeated use of the Levi enlargement lemma, we may find a pseudoline joining P to each point of the form $L_i \cap L_j$ to which it is not yet joined, arriving finally at an arrangment $\bar{\mathscr{A}}$ with the properties:

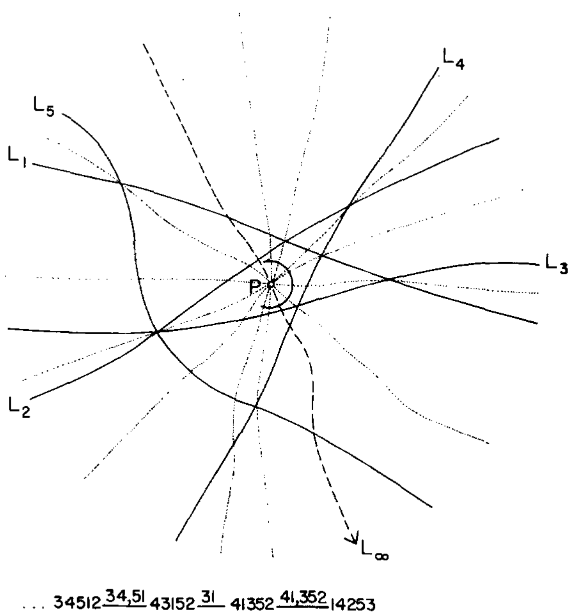
(i) $\mathscr{A} \subset \bar{\mathscr{A}}$;

(ii) each member of $\bar{\mathscr{A}} \backslash \mathscr{A}$ passes through P and through at least one point of the form $L_i \cap L_j$;

(iii) each $L_i \cap L_j$ lies on some member of $\bar{\mathscr{A}} \backslash \mathscr{A}$.

Such an arrangement $\bar{\mathscr{A}}$ will be called a *P-augmentation of* $\mathscr{A}$.

Just as the choice of a "vertical point at infinity" and a "left-to-right" direction determines a sequence asociated to an arrangment of lines, the same holds for an arrangement $\mathscr{A}$ of pseudolines as soon as we choose a P-augmentation $\bar{\mathscr{A}}$ as well as a "positive" direction of rotation around P: Let us copy the construction of the allowable sequence associated to an arrangement of lines. Choose a pseudoline "L_∞" passing through P but through no other intersection of members of $\bar{\mathscr{A}}$, with the help of Levi, moving it away from intersections if necessary (see Fig. 6). Beginning at P, if we traverse L_∞ in some direction, we determine a "starting" permutation of the numbers $1, \ldots, n$ according to the order in which we cross the pseudolines $L_1, \ldots, L_n$. If we now turn around P in the chosen positive direction, we encounter the members of $\bar{\mathscr{A}} \backslash \mathscr{A}$ in a definite (cyclic) order, and if one of them, say, L_i', passes through the intersection of several members of $\mathscr{A}$, say, $L_{i_1} \cap \cdots \cap L_{i_r}$, we make the corresponding "move" in the preceding permutation. (If L_i' passes through *several* intersections, the move consists, of course, of the corresponding simultaneous switches.) We now wish to show that these "moves" are possible within the context of an allowable



$\ldots\ 34512 \xrightarrow{34,51} 43152 \xrightarrow{31} 41352 \xrightarrow{41,352} 14253$

$\xrightarrow{42} 12453 \xrightarrow{12} 21453 \xrightarrow{45} 21543 \xrightarrow{15,43} 25134 \ldots$

FIGURE 6

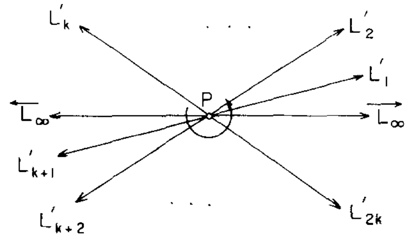


FIGURE 7

sequence; the following lemma does this, by giving an explicit construction for the terms of such a sequence:

LEMMA 2.4. *The "moves" given by reading off the order in which successive connecting pseudolines L_i' pass through the vertices of the arrangement $\mathscr{A}$, and continuing all around P, are those of an allowable sequence.*

Proof. Let us first renumber the directed pseudolines L_i', and their oppositely directed counterparts—call them L_{k+i}'—so that the P-pencil reads

$$\ldots, \vec{L}_\infty, L_1', \ldots, L_k', \overleftarrow{L}_\infty, L_{k+1}', \ldots, L_{2k}', \vec{L}_\infty, \ldots,$$

as in Fig. 7. For each $i = 1, \ldots, k$ draw two new directed pseudolines, L_i'' and L_i''', also passing through P and lying on opposite sides of L_i', in the (positive) order L_i'', L_i', L_i''' (see Fig. 8), and choose them "close" enough to L_i' so that $\{L_1, \ldots, L_n, L_\infty, L_1', \ldots, L_k', L_1'', \ldots, L_k'', L_1''', \ldots, L_k'''\}$ is still an arrangement. Again, let L_{k+i}'' (resp. L_{k+i}''') be L_i'' (resp. L_i''') with the opposite direction. Then if we read off the points $L_i' \cap \{L_1, \ldots, L_n\}$ going from P to P, for each i in the order $\ldots$, $1, \ldots, k$, $k+1, \ldots, 2k$, $1, \ldots$, we get the sequence of *moves* in question, while if we read off—for each i—the order in which L_i''

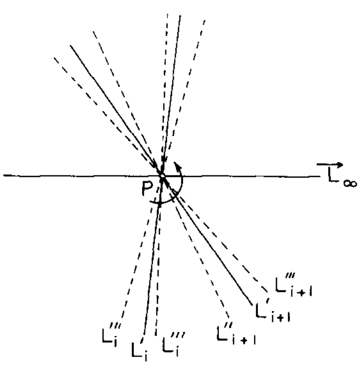


FIGURE 8

(resp. L_i''') meets $\{L_1,\ldots,L_n\}$, we get a permutation Π_i'' (resp. Π_i''') of $[1,n]$, and it is clear that the move corresponding to L_i' takes Π_i'' to Π_i'''. All that remains to be shown is that $\Pi_i''' = \Pi_{i+1}''$ for each i, modulo $2k$. But if we straighten the P-pencil (with L_∞ thought of as the line at infinity), this follows immediately from the well-known fact that if the opposite corners of a square are joined by paths lying inside the square, these paths must cross (see Fig. 9).

Now we can use the fact that a sequence is *uniquely* determined by its sequence of unordered switches (Corollary 1.13 or [9, Theorem 1.41]) to get:

COROLLARY 2.5. *A P-augmentation of $\mathscr{A}$, together with a choice of "positive" direction around P, canonically determines an allowable sequence whose moves correspond to the crossings of the members of $\mathscr{A}$; if the opposite direction around P is chosen as the "positive" one, the sequence is reversed.*

Remark 2.6. We could equally well have shown that the "moves" defined by the P-augmentation $\{L_1',\ldots,L_k'\}$ satisfy the three conditions of [7, Theorem 1], and hence come from an allowable sequence by that theorem, which—by [9, Theorem 1.4]—must be unique. However, the proof above actually shows what the terms of the allowable sequence are, namely, the permutations of $[1,n]$ determined by the order in which any choice of consistently directed pseudolines through P *in between the pseudolines* L_i' cross $\{L_1,\ldots,L_n\}$, just as in the case of an arrangement of *lines*.

PROPOSITION 2.7. *Varying the P-augmentation, or varying P within σ, produces an allowable sequence semispace-equivalent to the original one.*

Proof. Let us first keep P fixed and change the P-pencil $\{L_1',\ldots,L_k'\}$ to a new one, $\{\bar{L}_1',\ldots,\bar{L}_m'\}$. As long as we do not change the "positive" direction

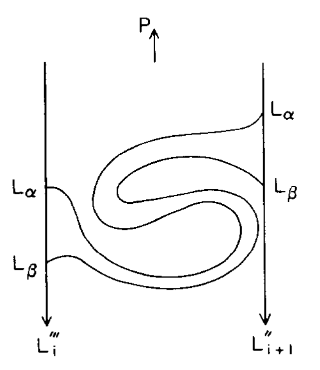


FIGURE 9

around P, it is clear that the cyclic order of the *directed* pseudolines ..., L'_1,..., L'_k, L'_{k+1},..., L'_{2k}, L'_1,... around P is the same as the cyclic order of their intersections with any fixed $L_i \in \mathscr{A}$ along L_i, as soon as we direct L_i in the appropriate way. (Notice that a direction around P induces canonically a direction along every pseudoline in $\mathbf{P}^2$ not passing through P.) It follows that the local sequence of ordered switches of any index i will be identical for the two P-augmentations, which gives the first statement by (iv) $\Rightarrow$ (i) of Theorem 1.7. On the other hand, if we move P to P' in the same cell σ, *and choose the corresponding positive direction around* P' (which makes sense since σ is orientable), it is clear that any P-pencil can be moved to a P'-pencil with only the portion of the L'_i lying within σ altered, and their cyclic order around P' the same as around P, namely, by a homeomorphism of σ to itself which extends to the identity on the rest of $\mathbf{P}^2$ (including $\partial\sigma$) and which maps P to P'. Hence the second statment follows.

It is not hard to see that if, instead of taking the P-augmentation of a pseudoline arrangement $\mathscr{A}$ with $P \in \sigma$, we replace σ by a cell σ' adjacent to it, say, separated by a pseudoline L_k, and take a P'-augmentation for some $P' \in \sigma'$, then the local L_i-sequences for $i \neq k$ do not change, while that of L_k gets reversed. This suggests the following:

DEFINITION 2.8. If Σ and Σ' are allowable n-sequences such that for each i, $1 \leqslant i \leqslant n$, the local sequence of unordered switches coming from Σ agrees with that coming from Σ', *except possibly for a reversal of direction*, we call Σ and Σ' *locally equivalent.*

For example, the sequences

$$\ldots 12345 \xrightarrow{12,45} 21354 \xrightarrow{135} 25314 \xrightarrow{25,14} 52341 \xrightarrow{234} 54321 \ldots$$

and

$$\ldots 35124 \xrightarrow{12} 35214 \xrightarrow{52,14} 32541 \xrightarrow{54} 32451 \xrightarrow{324} 42351 \xrightarrow{351} 42153 \ldots$$

are locally equivalent; notice that the 2-, 3-, and 5-sequences agree, while the 1- and 4-sequences are reversed.

Thus local equivalence is strictly coarser than semispace-equivalence, and its natural geometric role is seen in the following result:

THEOREM 2.9. *If $\mathscr{A}$ and $\mathscr{A}'$ are pseudoline arrangements, with $\mathscr{A}$ not a pencil, and Σ and Σ' are any allowable sequences corresponding to $\mathscr{A}$ and $\mathscr{A}'$ (respectively), then $\mathscr{A}$ and $\mathscr{A}'$ are isomorphic if and only if Σ and Σ' are locally equivalent.*

Proof. Suppose first that $\mathscr{A}$ and $\mathscr{A}'$ are isomorphic. Then since, under some self-homeomorphism f of $\mathbf{P}^2$, each $L_i \in \mathscr{A}$ gets mapped to $L_i' \in \mathscr{A}'$, and since f preserves (or reverses) the cyclic order of points on a simple closed curve, it is immediate that the local i-sequence in Σ is either the same as in Σ', or its reverse.

For the converse, we use induction on $|\mathscr{A}| = |\mathscr{A}'| = n$. If it is impossible to find a member of $\mathscr{A}$ whose removal leaves an arrangement $\mathscr{A}_0$ every member of which has at least three crossings occurring on it, then $\mathscr{A}$ must be a near-pencil (i.e., every pseudoline but one passes through a common point), or else a simple arrangement of three or four pseudolines, and in each of these cases the assertion is easy to verify directly. Otherwise, let us permute the indices so that the removal of L_n from $\mathscr{A}$ leaves an arrangement $\mathscr{A}_0$ every member of which carries at least three crossings. The condition "L_n crosses L_i at the intersection of L_i and $L_{j_1},..., L_{j_k}$" identifies a well-defined 0-cell on L_i where this crossing takes place, and the condition "L_n crosses L_i between the intersection of L_i and $L_{j_1},..., L_{j_k}$ and the intersection of L_1 and $L_{j_{k+1}},..., L_{j_m}$" identifies a well defined 1-cell on L_i where this crossing takes place, since L_i has at least three crossings occurring on it. By induction hypothesis, there is an isomorphism f_0: $\mathscr{A}_0 \to \mathscr{A}_0'$, where $\mathscr{A}_0' = \mathscr{A}' \backslash \{L_n'\}$. Then the equivalent position of the index n in the local i-sequences coming from Σ and Σ' implies that L_n and L_n' cut L_i and L_i' (resp.) along corresponding 0- or 1-cells. Since, by Proposition 2.1, any 2-cell of $\mathscr{A}_0$ which is entered by L_n is cut in a unique way into two 2-cells of $\mathscr{A}$, and similarly for $\mathscr{A}_0'$ and $\mathscr{A}'$, it is then a trivial matter to extend f_0 to an isomorphism f: $\mathscr{A} \to \mathscr{A}'$.

Remark 2.10. Alexander *et al.* define the notion of an "order tableau" in [1], which bears some resemblance to our family of local sequences, and associate such an order tableau to any simple arrangement of pseudolines in the euclidean plane. Essentially, the order tableau of an arrangement $\mathscr{A} = \{L_1,..., L_n\}$ records the order in which each L_i meets the pseudolines L_j, $1 \leqslant j < i$. They then prove that two simple arrangements are isomorphic if and only if (after renumbering) their order tableaux agree. Their result would not hold in $\mathbf{P}_2$, however, even for simple arrangements, as the example in Fig. 10 shows.

We can now characterize isomorphism of *marked* arrangements in terms of their associated sequences:

COROLLARY 2.11. *If* $(\mathscr{A}, \sigma)$ *and* $(\mathscr{A}', \sigma')$ *are isomorphic marked arrangements, then any allowable sequence coming from* $(\mathscr{A}, \sigma)$ *is semispace-equivalent to any allowable sequence coming from* $(\mathscr{A}', \sigma')$. *Conversely, if a sequence coming from* $(\mathscr{A}, \sigma)$ *is semispace-equivalent to one coming from* $(\mathscr{A}', \sigma')$, *then* $(\mathscr{A}, \sigma)$ *is isomorphic to* $(\mathscr{A}', \sigma')$.

Proof. The first implication is immediate from Proposition 2.7, as soon

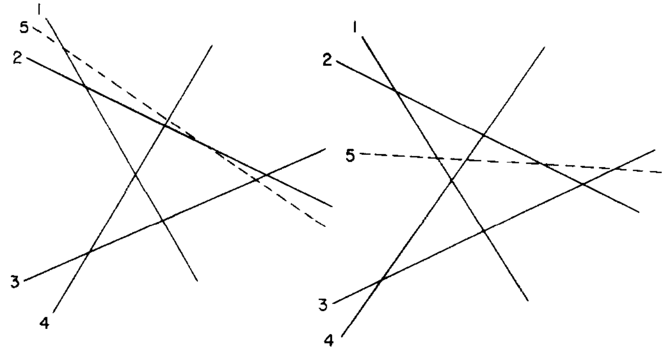


FIGURE 10

as we observe that the isomorphism can be realized by a homeomorphism of $\mathbf{P}^2$ to itself, and this takes a P-augmentation of $(\mathscr{A}, \sigma)$ to a P'-augmentation of $(\mathscr{A}', \sigma')$ for some point $P' \in \sigma'$. Conversely, since by Theorem 2.9 we already know that $\mathscr{A}$ and $\mathscr{A}'$ are isomorphic, i.e., that there is a homeomorphism f: $\mathbf{P}^2 \to \mathbf{P}^2$ such that $f(L_i) = L_i'$ for $1 \leqslant i \leqslant n$, where $\mathscr{A} = \{L_1, \ldots, L_n\}$ and $\mathscr{A}' = \{L_1', \ldots, L_n'\}$, it remains only to show that $f(\sigma) = \sigma'$. But this is clear, since if σ and σ' were noncorresponding cells, any sequence derived from $(\mathscr{A}', \sigma')$ would contain at least one local sequence in the reverse order from the corresponding local sequence coming from $(\mathscr{A}, \sigma)$ (see the remark preceding Definition 2.8), which would violate the hypothesis of semispace-equivalence.

Remark 2.12. Theorem 2.9 and Corollary 2.11 can also be derived from the Folkman–Lawrence representability theorem [6] on the relation between acyclic oriented matroids and arrangements of pseudohemispheres, with the help of Theorem 1.7 above.

3. Generalized Configurations

Just as arrangements of pseudolines constitute a natural generalization of arrangements of lines, we can generalize configurations of points in a corresponding way to yield what is perhaps the most natural geometric way of looking at allowable sequences. This procedure has been suggested in [7] and [11],[1] and we now proceed to describe it in detail.

DEFINITION 3.1. Suppose $(\mathscr{C}, \mathscr{A})$ is a pair consisting of a configuration $\mathscr{C} = \{P_1, \ldots, P_n\}$ of points in $\mathbf{P}^2$ and an arrangement $\mathscr{A} = \{L_1, \ldots, L_k, L_\infty\}$ of pseudolines, with L_∞ a *directed* pseudoline, such that

[1] For a parallel development involving oriented matroids, see [3].

(i) every pair of points of $\mathscr{C}$ lies on some member of $\mathscr{A}$;

(ii) each of $L_1,..., L_k$ contains at least two points of $\mathscr{C}$;

(iii) L_∞ contains no P_i.

Then $(\mathscr{C}, \mathscr{A})$ is called a ***generalized configuration of points***, L_∞ is called the ***pseudoline at infinity***, and the remaining members of $\mathscr{A}$ are called the ***connecting pseudolines***.

DEFINITION 3.2. Given a generalized configuration $(\mathscr{C}, \mathscr{A})$. If we arbitrarily choose a direction transversal to L_∞ as the "positive" one along L_i at a crossing $L_i \cap L_\infty$, and move in the positive direction along L_∞ until we return to the same point, this "positive" transversal direction will reverse, since $\mathbf{P}^2$ is nonorientable. Hence if we move *twice* along L_∞, we will recover the original "positive" transversal direction. Thus we get a periodic sequence of orientations of the pseudolines $L_1,..., L_k$, which breaks up into half-periods with the property that each orientation is reversed a half-period later. For simplicity let L_{k+i}, $1 \leqslant i \leqslant k$, be L_i with the opposite orientation, and renumber $L_1,..., L_{2k}$ so that as we traverse L_∞ we meet them in the cyclic order $...L_1,..., L_k, L_{k+1},..., L_{2k}, L_1,...$ (i.e., the orientations of $L_1,..., L_k$ "agree," etc.); of course we may meet several of the L_i simultaneously. Let M_i, $1 \leqslant i \leqslant 2k$, be the sequence in which those of the points $P_1,..., P_n$ which lie on L_i do so, starting and ending at $L_i \cap L_\infty$; each M_i is thus an ordered r-tuple for some r, $2 \leqslant r \leqslant n$, and M_{k+i} is the reverse of M_i for each i. The periodic sequence

$$..., M_1,..., M_k, M_{k+1},..., M_{2k}, M_1,...$$

is called the sequence of *moves* of the generalized configuration $(\mathscr{C}, \mathscr{A})$.

EXAMPLE 3.3. The sequence of moves coming from the generalized configuration in Fig. 11 is

$$...; 24, 35; 31; 251; 23, 41; 45; 43; 42, 53; 13; 152; 14, 32; 54; 34;....$$

(Notice that if several of the pseudolines L_i cross L_∞ at the same point, we amalgamate the "moves" along each of them into a single move, as above: 24 and 35 occur in the same move (as well as 42 and 53), and similarly for 23 and 41.)

In [7, Theorem 1] it is proven that the sequence of "moves" coming from such a pair $(\mathscr{C}, \mathscr{A})$ is actually the sequence of ordered moves of some allowable sequence Σ, and in [9, Theorem 1.4] that Σ is uniquely determined. (The former is proven only in the case where each move consists of only *one* substring of $[1, n]$, i.e., where distinct pseudolines L_i cross L_∞ at

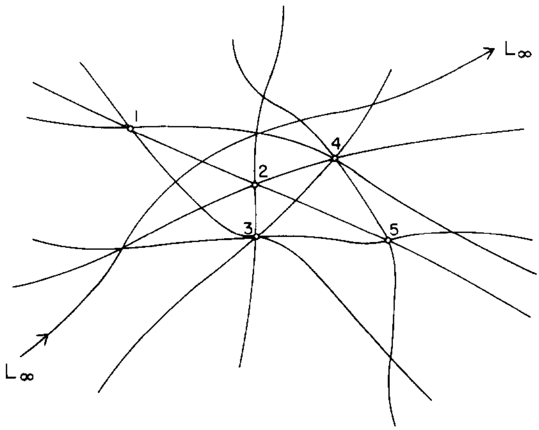


FIGURE 11

distinct points, but the proof extends immediately to the more general situation considered here.) Rather than repeating the proof, let us merely recall, in the case of the example above, how to find the corresponding 5-sequence Σ.

To find the term of Σ immediately preceding the move 24, 35 we consider the half-period of moves beginning with 24, 35 and ending with 43, and notice that if we interpret each move as defining an ordering of the numbers 1,..., 5:

$$2 \prec 4,\ 3 \prec 5,\ 3 \prec 1,\ 2 \prec 5 \prec 1,\ 2 \prec 3,\ 4 \prec 1,\ 4 \prec 5,\ 4 \prec 3,$$

then these separate pieces of information are all consistent, and in fact determine the order

$$2 \prec 4 \prec 3 \prec 5 \prec 1,$$

which gives 24351 as the term of Σ occurring just before the move 24,35. Applying this move then gives the next term of Σ, namely, 42531, and the move which follows, namely, 31, is automatically compatible with this permutation. In this way we construct the entire sequence

$$\ldots 24351 \xrightarrow{24,35} 42531 \xrightarrow{31} 42513 \xrightarrow{251} 41523 \xrightarrow{41,23} 14532 \xrightarrow{45} 15432$$

$$\xrightarrow{43} 15342 \xrightarrow{53,42} 13524 \xrightarrow{13} 31524 \xrightarrow{152} 32514 \xrightarrow{32,14} 23541$$

$$\xrightarrow{54} 23451 \xrightarrow{34} 24351 \ldots.$$

We can now state

DEFINITION 3.4. The unique n-sequence which gives rise to the sequence of moves of the generalized configuration $(\mathscr{C}, \mathscr{A})$ is called the *sequence of permutations associated to* $(\mathscr{C}, \mathscr{A})$.

Remark 3.5. If $\mathscr{C}$ is a configuration of points in $\mathbf{E}^2$ and $\mathscr{A}$ consists of all the (straight) lines joining two or more of the points of $\mathscr{C}$ plus the line at infinity in a projective completion of $\mathbf{E}^2$, with the "counterclockwise" direction chosen as "positive," then the sequence of permutations associated to $(\mathscr{C}, \mathscr{A})$ is precisely the sequence associated to the configuration $\mathscr{C}$, in the sense of Definition 1.1.

Remark 3.6. All of the geometric notions pertaining to a configuration of points mentioned in Remark 1.3 make sense for a generalized configuration $(\mathscr{C}, \mathscr{A})$; e.g., $ij//km$ means that the connecting pseudolines in question meet on L_∞, a semispace is the set of all points in $\mathscr{C}$ lying in some "halfplane" determined by L_∞ and a pseudoline L' such that $\mathscr{A} \cup \{L'\}$ is an arrangement, etc. Furthermore, it is clear, just as it was for configurations, that these geometric properties are described by the correspondingly named properties of the associated sequence (see Remark 1.4). Corollary 1.10 therefore extends immediately to generalized configurations, and we shall call two generalized configurations *semispace-equivalent* if they satisfy the conditions of Corollary 1.10. (Notice, therefore, that to say that $(\mathscr{C}, \mathscr{A})$ is semispace-equivalent to $(\mathscr{C}', \mathscr{A}')$ means, by virtue of condition (iv), that the connecting pseudolines around correponding points correspond in their cyclic order.) Finally, we shall say that two generalized configurations are *locally equivalent* if their associated sequences are.

LEMMA 3.7. *If Σ and Σ' are locally equivalent sequences, such that for every extreme point i of Σ the local i-sequences in Σ and in Σ' agree* without reversal, *then Σ and Σ' are semispace-equivalent.*

Proof. Suppose Σ is not semispace-equivalent to Σ'. Then there is a (nonextreme) point j of Σ such that the j-sequences in Σ and in Σ' go in opposite directions. Choose three extreme points, i_1, i_2, i_3, and delete all the indices except these and j. Then i_1, i_2, i_3 remain extreme points, j may or may not remain nonextreme, and it is still true that the i_1-, i_2-, and i_3-sequences in the restrictions Σ_0 and Σ_0' of Σ and Σ' (resp.) agree, while the j-sequences are reversed. By renumbering, we are reduced to showing that in any two extensions of

$$123 \xrightarrow{12} 213 \xrightarrow{13} 231 \xrightarrow{23} 321$$

to a sequence on $\{1, 2, 3, 4\}$ in which the 1-, 2-, and 3-sequences *still* agree, and in which 1, 2, and 3 are still extreme points, the 4-sequences agree as well, and this follows immediately by a simple enumeration.

Here is the counterpart, for local equivalence, of (iv) $\Rightarrow$ (vi) of Theorem 1.7:

THEOREM 3.8. *If Σ and Σ' are locally equivalent sequences, there is a succession of sequences $\Sigma = \Sigma_0, \Sigma_1, ..., \Sigma_n = \Sigma'$ (or $\tilde{\Sigma}'$), all locally equivalent, such that each pair Σ_i, Σ_{i+1} differ in the direction of at most one local sequence.*

Proof. If Σ and Σ' are locally equivalent but not semispace-equivalent, fix an index i_0 whose local sequence goes the same way in both. By Lemma 3.7, there is an extreme point i_1 for Σ whose local sequences are opposite. By elementary transformations on Σ, as in the proof of Theorem 1.7, (iv) $\Rightarrow$ (vi), we can get all the switches of the form $i_1 S$ in Σ to be consecutive. Say the sequence now looks like

$$... i_1 S_1 \cdots S_k \xrightarrow{i_1 S_1} S_1 i_1 S_2 \cdots S_k ... S_1 \cdots S_{k-1} i_1 S_k \xrightarrow{i_1 S_k} S_1 \cdots S_k i_1 ....$$

Then we can reverse the local i_1-sequence, without affecting any of the other local sequences, by replacing this portion of the sequence by

$$... S_1 \cdots S_k i_1 \xrightarrow{S_k i_1} S_1 \cdots S_{k-1} i_1 S_k ... S_1 i_1 S_2 \cdots S_k \xrightarrow{S_1 i_1} i_1 S_1 \cdots S_k ...,$$

the corresponding portion a half-period later by the same terms, each written in reverse order, and changing each remaining term in the sequence by moving i_1 either from the beginning to the end, or from the end to the beginning. Downward induction on the number of reversed indices then finishes the proof.

Remark 3.9. Let us interpret the procedure in Theorem 3.8 geometrically, both for arrangements and for generalized configurations. As noted previously, if we derive a sequence Σ from a marked arrangement ($\mathscr{A}$, σ), and then choose a cell σ' across one pseudoline $L_i \in \mathscr{A}$ from σ, any sequence Σ' derived from ($\mathscr{A}$, σ') will be locally equivalent to Σ and the local sequences (of unordered switches) in Σ and Σ' will all agree, except for the i-sequence, which will be reversed. Thus the procedure in Theorem 3.8 amounts merely to following a path from one cell, σ, of A, to another, σ', which crosses each member of $\mathscr{A}$ at most once. In terms of generalized configurations ($\mathscr{C}$, $\mathscr{A}$), on the other hand, what we are doing is moving the pseudoline at infinity, L_∞, stepwise to a new location, L'_∞, with each reversal of a local sequence amounting to moving L_∞ across one extreme point of $\mathscr{C}$. (This procedure can even be carried out continuously, in fact, and yields a kind of isotopy for locally equivalent generalized configurations.)

Thus for generalized configurations we get:

COROLLARY 3.10. *$(\mathscr{C}, \mathscr{A})$ and $(\mathscr{C}', \mathscr{A}')$ are locally equivalent if and only if they become semispace-equivalent when the pseudoline at infinity in $\mathscr{A}$ is moved to an appropriate position.*

Proof. If $(\mathscr{C}, \mathscr{A})$ and $(\mathscr{C}', \mathscr{A}')$ are semispace-equivalent, then *a fortiori* they are locally equivalent; since the position of the pseudoline at infinity in $\mathscr{A}$ has no effect on the local sequences coming from $\mathscr{C}$, except possibly to reverse their direction, the sufficiency of the condition follows. For the necessity, just use Theorem 3.8 and Remark 3.9.

4. REALIZABILITY AND "PSEUDO-REALIZABILITY"

Since not every pseudoline arrangement is stretchable [18, 21], it follows from Theorem 2.9 that not every allowable sequence can be realized by lines, or even "semispace-realized," or even "locally realized" for that matter. Hence, by Theorem 1.6 of [9], there are generalized configurations of points which are not combinatorially, or semispace-, or locally equivalent to any actual configurations of points. (A minimal nonrealizable n-sequence ($n = 5$) is given in [8, Theorem 3.3], while a minimal n-sequence which is not semispace- or locally realizable ($n = 9$) can be derived from [17, Fig. 3.3].)

On the other hand, we have:

THEOREM 4.1. *If Σ is an n-sequence, there is an arrangement $\mathscr{A}$ of n pseudolines and a P-augmentation $\bar{\mathscr{A}}$ of $\mathscr{A}$ such that Σ is the sequence associated to $\bar{\mathscr{A}}$.*

Proof. The proof follows the construction given in [7, Theorem 2] with a more restricted definition of "allowable sequence"; we realize Σ by a sort of "wiring diagram," as follows: Suppose

$$\Sigma = ..., \Pi_1, ..., \Pi_N, \bar{\Pi}_1, ..., \bar{\Pi}_N, \Pi_1, ...$$

as in Definition 1.2, and suppose M_i is the move from Π_i to Π_{i+1}. We start drawing n horizontal "wires," labeled from top to bottom in the order given by Π_1. We then cross the wires appearing in each substring being switched in M_1, and continue drawing the wires horizontally. Then apply M_2, and so on, until we have made the first N moves; the wires are then in the order given by $\bar{\Pi}_1$. (Notice that because each M_i reverses *substrings* of Π_i, each bunch of wires we cross consists of *adjacent* wires; hence there is no obstruction to drawing the picture.) Finally, we extend the beginnings of the wires in a monotone sequence of directions, and the ends oppositely, so as to get an arrangement $\mathscr{A}$ of pseudolines which we can think of as lying in $\mathbf{P}^2$. (Figure 12 shows the "wiring diagram" realization of the sequence

$$\ldots 12345 \xrightarrow{123,45} 32154 \xrightarrow{15} 32514 \xrightarrow{14} 32541 \xrightarrow{25} 35241$$
$$\xrightarrow{35,24} 53421 \xrightarrow{34} 54321\ldots.)$$

Now let P be the "vertical point at infinity," and draw a vertical line through each crossing. This gives a P-augmentation $\bar{\mathscr{A}}$ of $\mathscr{A}$, whose associated sequence is clearly Σ.

The proof of (iv) $\Rightarrow$ (v) of Theorem 1.7, which we previously postponed, now follows easily as a corollary:

COROLLARY 4.2. *Let Σ and Σ' be two nontrivial allowable n-sequences. Then the following are equivalent*:

(iv) *Σ and Σ' (or else $\tilde{\Sigma}'$) have the same local sequence of ordered switches for each i, $1 \leqslant i \leqslant n$;*

(v) *Σ and Σ' (or else $\tilde{\Sigma}'$) have the same local sequence of unordered switches for each i, $1 \leqslant i \leqslant n$.*

Proof. It is clear that (iv) implies (v). Conversely, if (v) holds, then letting $\bar{\mathscr{A}}$ (resp. $\bar{\mathscr{A}}'$) be an augmented pseudoline realization of Σ (resp. Σ'), we see from Theorem 2.9 that the underlying pseudoline arrangements $\mathscr{A}$ and $\mathscr{A}'$ are isomorphic. ***But this isomorphism must map the distinguished cell σ of $\bar{\mathscr{A}}$ into the distinguished cell σ' of $\bar{\mathscr{A}}'$***, since otherwise at least one local sequence would reverse direction from Σ to Σ'. Hence by Corollary 2.11 condition (iv) holds.

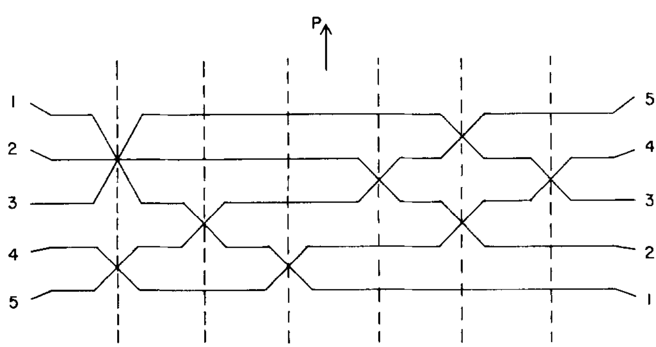


FIGURE 12

Remark 4.3. In [21] Ringel gives a purely combinatorial description of the cell complexes associated to all simple arrangements (no three concurrent) of pseudolines in the plane, and in [17] Grünbaum remarks that no such description is known in the general case, i.e., without the assumption of simplicity. The desired description now follows immediately from our association of an allowable sequence to each pseudoline arrangement, and from the fact that every allowable sequence may be realized by a pseudoline arrangement.

Here is a combinatorial description of the cell complex $\Gamma(\mathscr{A})$ associated to an arrangement $\mathscr{A}$ of pseudolines in $\mathbf{P}^2$, in terms of a sequence Σ associated to $\mathscr{A}$; its truth is most easily verified by staring at Fig. 12:

A 0-cell V of $\Gamma(\mathscr{A})$ corresponds to a switch $S = i_1 \cdots i_k$ occurring in some move of Σ (as well as to $\bar{S}$);

a 1-cell E of $\Gamma(\mathscr{A})$ corresponds to a pair of switches of Σ which occur consecutively in the local sequence of some (necessarily unique) index i (as well as to the corresponding pair a half-period later);

a 2-cell F of $\Gamma(\mathscr{A})$ corresponds to a complementary pair of semispaces of Σ (including the pair $\{\varnothing, \{1,..., n\}\}$)—namely, a point P lying in such a cell determines which pseudolines are "above" and "below" P, and such a pair of complementary semispaces in turn determines a unique 2-cell, by Lemmas 4.4 and 4.5 of [11];

V is incident to E if it corresponds to one of the two switches corresponding to E;

E is incident to F if, in every term of Σ occurring between the two consecutive switches in the local i-sequence which corresponds to E, the index i is at the boundary between the two complementary semispaces corresponding to F.

We can now use the dualizing procedure developed in [7] to realize an allowable sequence in a way closer in spirit to the original geometric situation from which such sequences arose in the first place, namely, by a generalized configuration of points:

THEOREM 4.4. *Every allowable sequence can be realized by a generalized configuration.*

Proof. [Let us follow an example through the steps of the realization process.] Given a sequence Σ [for example,

$$\Sigma\colon ...12345 \xrightarrow{23} 13245 \xrightarrow{13,24} 31425 \xrightarrow{25} 31452$$
$$\xrightarrow{145} 35412 \xrightarrow{35,12} 53421 \xrightarrow{34} 54321...],$$

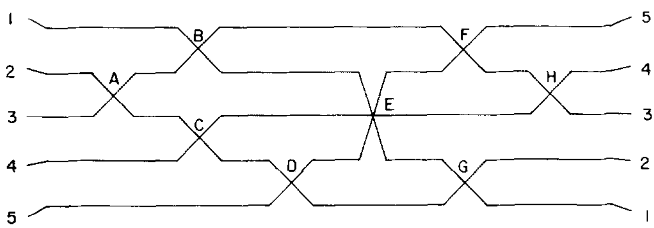


FIGURE 13

we first realize a half-period of it by a wiring diagram [as in Fig. 13], labeling all the switches from left to right, and simultaneous switches from top to bottom. Next, we use the Levi Enlargement Lemma repeatedly to join all those pairs of switches not yet joined, using vertical lines for simultaneous switches [as in Fig. 14]. Now read off the sequence Σ^* associated to this generalized configuration, or at least the half-period of it beginning with the vertical direction, by reading off the sequence of moves starting with the vertical switches and proceeding counterclockwise. [In the example, we get

$$\Sigma^*\colon \ldots ABCDEFGH \xrightarrow{BC,FG} ACBDEGFH \xrightarrow{BD} ACDBEGFH$$

$$\xrightarrow{BEG} ACDGEBFH \xrightarrow{ACDG} GDCAEBFH \xrightarrow{AE} GDCEABFH$$

$$\xrightarrow{ABFH} GDCEHFBA \xrightarrow{CEH} GDHECFBA \xrightarrow{CF} GDHEFCBA$$

$$\xrightarrow{DH} GHDEFCBA \xrightarrow{DEF} GHFEDCBA \xrightarrow{GH} HGFEDCBA \ldots]$$

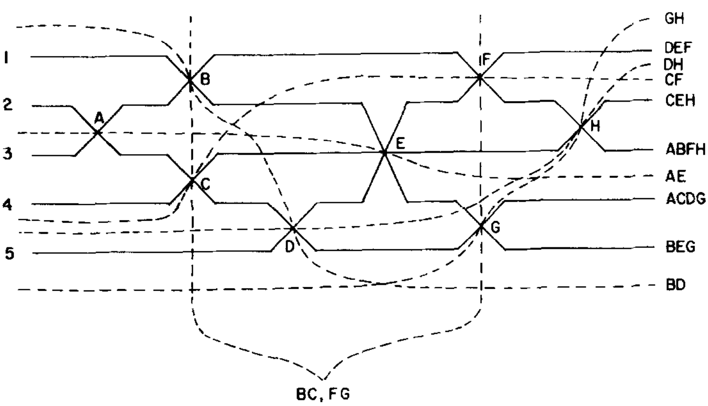


FIGURE 14

Finally, realize *this* sequence by a wiring diagram [as in Fig. 15]. Since all the original pseudolines [1, 2, 3, 4, 5 in the example] correspond to some of the moves in the sequence Σ^*, and these occur in the same order [*BEG*, *ACDG*, *ABFH*, *CEH*, *DEF*, respectively, in the example], we can label the corresponding vertices in this last arrangement accordingly [as in Fig. 15], and we have produced a generalized configuration which realizes Σ, provided we take L_∞, the pseudoline at infinity, to be the vertical line passing through all the left-most crossings.

Recall that in [9] we showed that an allowable sequence is realizable by a configuration of points if and only if it is realizable by an arrangement of lines. We can therefore summarize Theorems 2.9, 4.1, and 4.4 and Corollaries 2.11 and 3.10, as well as this fact, in the following comprehensive statement:

Main result of Sections 2–4. Every allowable sequence of permutations can be realized by a generalized configuration of points as well as by an arrangement of pseudolines. This association induces a bijection α between the set of generalized configurations modulo local equivalence and the set of pseudoline arrangements modulo isomorphism, as well as a bijection β between the set of generalized configurations modulo semispace-equivalence and the set of marked pseudoline arrangements modulo isomorphism. Under both α and β, the equivalence classes consisting of (genuine) configurations correspond to the isomorphism classes of straight-line arrangements.

5. Some Open Problems

PROBLEM 5.1. A plane configuration of points determines a distinguished family of subsets, namely, its semispaces. One may therfore ask: Given a family $\mathscr{H}$ of subsets of a set S of n elements, when can S be

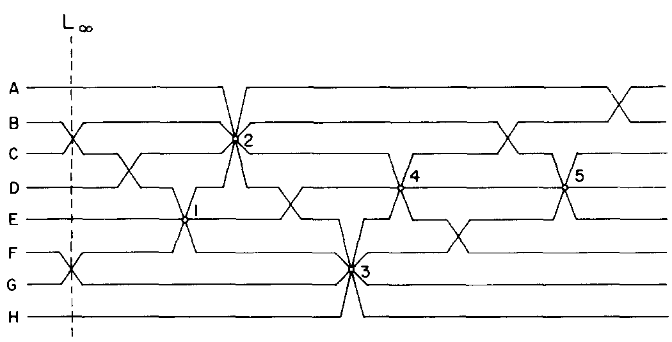


FIGURE 15

embedded in the plane in such a way that $\mathscr{H}$ becomes its family of semispaces? Our work provides a combinatorial answer to the same question about *generalized* configurations (as well as about ordinary configurations up to $n=8$, by virtue of [8]), namely: if and only if $\mathscr{H}$ is the family of initial segments of some allowable n-sequence. It is even not difficult to write down an algorithm for testing this, but the original geometric question seems much harder; it is equivalent, in fact, to the "coordinatizability problem" for oriented matroids of rank 3, and to the stretchability problem for pseudoline arrangements, both of which have been open for some time. (See [8, Corollary 3.4] for further combinatorial necessary conditions for realizability, at least in the simple case.)

PROBLEM 5.2. A second problem which seems quite difficult is the isotopy problem. In terms of configurations, this asks: Given a configuration $\mathscr{C}$ semispace-equivalent to a configuration $\mathscr{C}'$, can $\mathscr{C}$ be deformed continuously into $\mathscr{C}'$ (or its reflection), with each intermediate configuration remaining within the same semispace-equivalence class? If we think of a configuration $\mathscr{C}$ of n points in $\mathbf{E}^2$ as represented by a single point P in $\mathbf{E}^{2n}$, this amounts to asking whether the set Γ in $\mathbf{E}^{2n}$ corresponding to an entire semi-equivalence class has only two components. (One may ask, moreover, whether each component of Γ is simply-connected, or even what its homology type is, in general.) Along these lines, let us mention one result which is not hard to see: even if Γ has only two components, these need not be convex! (Put in more concrete terms, it may be possible to pass from a configuration $\mathscr{C}$ to a semispace-equivalent configuration $\mathscr{C}'$ by a nonlinear isotopy, and yet impossible to do so linearly.) It may turn out that the isotopy problem has a positive solution if $\mathscr{C}$ and $\mathscr{C}'$ are restricted to be simple configurations, without having a positive solution in general. As mentioned above, Theorem 1.7, (i) $\Rightarrow$ (vi), is a discrete version of an isotopy theorem; perhaps it may shed some light on the continuous problem (see [15]).

PROBLEM 5.3. There is the problem of counting configurations. If $N(n)$ (respectively, $\tilde{N}(n)$) represents the number of semispace-equivalence classes of numbered n-point configurations (respectively, generalized configurations), the problem is to find nontrivial bounds on N and $\tilde{N}$. We have shown elsewhere [14] that

$$\exp(cn \log n) \leqslant N(n) \leqslant \exp(cn^2 \log n)$$

and that

$$\exp(cn^2) \leqslant \tilde{N}(n) \leqslant \exp(cn^2 \log n).$$

These bounds, especially the ones for $N(n)$, need to be refined. One difficulty that arises in trying to count semispace-equivalence classes of configurations, or of generalized configurations, is that one representative of a semispace-equivalence class may admit certain extensions which another may not; this is illustrated in Fig. 8 on p. 233 of [8] (see footnote 1). On the other hand, R. Stanley [22] has recently proven a formula for the number of simple allowable sequences containing a given term, say, 123...n. Perhaps one can use this to derive reasonable bounds on the number of semispace-equivalence classes of simple configurations.

PROBLEM 5.4. Suppose we are given a "cluster of stars," with the center of each star labeled i ($1 \leqslant i \leqslant n$), and the lines radiating from point i labeled with all the indices $j \neq i$ and directed. (See Fig. 16a, where a simple 4-cluster is illustrated.) Under what conditions can one complete the picture to a generalized configuration, with each j-line radiating from point i connecting to the i-line radiating from point j in such a way that the directions match (see Fig. 16b)? This is an equivalent formulation of the following purely combinatorial problem: Given a set of candidates for the local sequences of ordered switches of the indices $1,..., n$, find necessary and sufficient conditions for the existence of a global n-sequence which puts them all together. (Clearly the *position* of the stars themselves plays no role: given two disjoint sets of closed disks $\Delta_1,..., \Delta_n$ and $\Delta_1',..., \Delta_n'$ in the plane, and a homeomorphism $f: \cup \Delta_i \to \cup \Delta_i'$, one can always extend f to a homeomorphism of the plane to itself.) A solution, besides being interesting in its own right, may also help with Problem 5.3.

PROBLEM 5.5. The procedure given in Theorem 4.4 for realizing an allowable sequence by a generalized configuration of points involves essen-

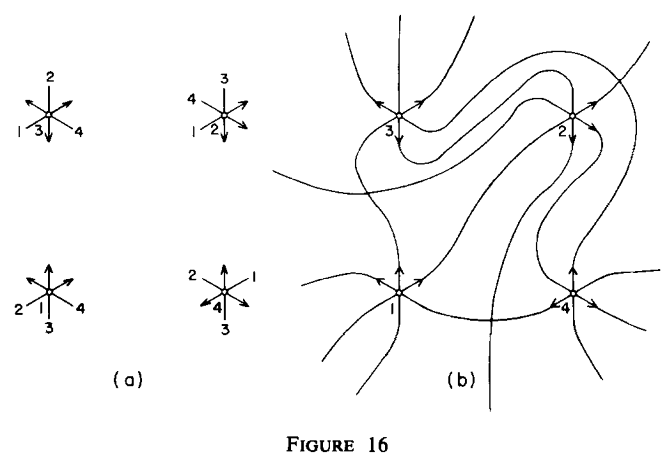


FIGURE 16

tially a "double-dualizing" process. We know of no way to achieve the same result directly; it would be interesting to have a procedure for doing this, for example, beginning with a "cluster of stars" (as in Problem 5.4) representing the local sequences.

PROBLEM 5.6. We have seen that allowable sequences provide an effective set of tools for investigating configurations and arrangements in $\mathbf{E}^2$ and $\mathbf{P}^2$ (see also the papers mentioned in the Introduction). What about higher dimensions? We need a device analogous to allowable sequences which will enable us to encode all the order properties of a configuration of points or arrangement of hyperplanes (or pseudohyperplanes) in such a way that it can be as easily manipulated as the allowable sequences we deal with here. Perhaps we should note, in this connection, that one of the tools we have found essential in the plane, in dealing with pseudoline arrangements and generalized configurations, namely, the Levi Enlargement Lemma, does not extend to higher dimensions [12]; on the other hand, it should not be difficult to extend the notion of allowable sequence in such a way as to permit at least *ordinary* configurations in higher dimensions to be studied combinatorially.

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